EDGE-COLOURED GRAPHS AND SWITCHING WITH S_m, A_m AND D_m

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Abstract. We consider homomorphisms and vertex colourings of m-edge-coloured graphs that have a switching operation which permutes the colours of the edges incident with a specified vertex. The permutations considered arise from the action of the symmetric, alternating and dihedral groups on the set of edge colours. In all cases, after studying the equivalence classes of m-edge-coloured graphs determined by the switching operation, we describe dichotomy theorems for the complexity of the vertex k-colouring problem and the problem of deciding the existence of a homomorphism to a fixed target H.

1. Introduction

It is often true that a result which holds for *2-edge-coloured graphs* (graphs with two disjoint edge sets) also holds for oriented graphs with essentially the same proof. On the other hand, Sen [13] has given examples of statements that hold for oriented graphs and not for 2-edge-coloured graphs. These examples may help to explain why a direct translation between the two families has not been found, and may not exist. Instead, Nešetřil and Raspaud [11] made a connection between them by defining (n, m)-mixed graphs, which have n disjoint arc sets and m disjoint edge sets. Oriented graphs are $(1, 0)$-mixed graphs, and 2-edge-coloured graphs are $(0, 2)$-mixed graphs. Results that hold for (n, m)-mixed graphs hold for both families just mentioned, and others. Conversely, if a theorem holds for both oriented graphs and 2-edge-coloured graphs, then there is evidence that perhaps a general theorem holds for (n, m)-mixed graphs.

The literature contains results about colourings and homomorphisms of *signed graphs*, 2-edge-coloured graphs that also have an operation that reverses the sets to which the edges incident with a specified vertex

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belong, see [3, 12, 13]. These results are similar to results for oriented graphs with an operation that reverses the arcs incident with a vertex (see [8]). Hence it may be true that there are general theorems of which these results are special cases.

As a step in the quest for such theorems, MacGillivray and Warren [10] consider m-edge-coloured graphs ($(0, m)$-graphs) together with a switching operation in which the elements of a group $\Gamma \leq S_m$ are used to permute the sets to which the edges incident with a specified vertex belong. They showed that some theorems from [2] and [8] about homomorphisms and colourings hold in this more general setting, and others hold when the group Γ is Abelian.

In this paper we consider the case when the group Γ is one of several non-Abelian groups, specifically the symmetric group S_m, where $m \geq 3$, the alternating group A_m, where $m \geq 4$, and the dihedral group D_m, where $m \geq 3$ (here D_m is the group of symmetries of a regular m-gon). We prove results that specify when a given m-edge-coloured graph G can be transformed by a sequence of switching operations using one of these groups so that it becomes isomorphic to a given m-edge-coloured graph H. We then consider the questions of when a given m-edge-coloured graph G can be transformed to that it has a homomorphism to a fixed m-edge-coloured graph H, or a vertex k-colouring for some fixed integer $k \geq 2$. In all cases we describe a dichotomy between the cases which are Polynomial and those which are NP-complete.

The colourings considered here are different than those considered by Zaslavsky [15]. His colourings are invariant under switching, whereas in our case switching may change the number of colours needed.

2. Preliminaries

For basic definitions in graph theory, see the text by Bondy and Murty [1]. We consider only finite, simple graphs and refer to them as graphs in the interest of brevity. For results about graph colourings and homomorphisms, see the book by Hell and Nešetřil [7]. For basic results in Abstract Algebra, see the text by Gallian [5].

Let $m \geq 1$ be an integer. A *m-edge-coloured graph* is an $(m+1)$-tuple

$$G = (V(G), E_0(G), E_1(G), \ldots, E_{m-1}(G)),$$

where

(i) $V(G)$ is a set of objects call *vertices*;

(ii) for $0 \leq i \leq m-1$, $E_i(G)$ is a set of unordered pairs of not-necessarily distinct vertices called the *edges of colour* i; and
(iii) $E_i(G) \cap E_j(G) = \emptyset$ when $i \neq j$.

When the context is clear, the vertex set is referred to as V and the edge sets as $E_0, E_1, \ldots, E_{m-1}$.

If G is an m-edge-coloured graph, then the *underlying graph of* G is the graph with vertex set V and edge set $E_0 \cup E_1 \cup \cdots \cup E_{m-1}$. An m-edge-coloured graph G can be regarded as being obtained from its underlying graph by assigning each edge one of the m available colours.

Let G an m-edge-coloured graph, and $\Gamma \leq S_m$. For $x \in V(G)$, the operation of *switching at* x *with respect to* $\gamma \in \Gamma$ transforms G into the m-edge-coloured graph $G^1 = G \cdot (x, \gamma)$ with vertex set $V(G)$ and edge sets

$$E_i(G^1) = (E_i(G) - \{xy : xy \in E_i(G)) \cup \{xz : xz \in E_j(G) \textit{ and } \gamma(j) = i\}.$$

The permutation γ acts on the colours of the edges incident with x: the edge xy belongs to $E_i(G^1)$ if and only if $xy \in E_j(G)$ and $\gamma(j) = i$.

Let $\Gamma \leq S_m$. Define the relation $\sim_\Gamma$ on the set of all m-edge-coloured graphs by $G \sim_\Gamma H$ if and only if there is a finite sequence $(x_1, \gamma_1), (x_2, \gamma_2), \ldots, (x_t, \gamma_t)$ where $x_i \in V$ and $\gamma_i \in \Gamma$, $1 \leq i \leq t$, such that if $G^0 = G$ and $G^i = G^{i-1} \cdot (x_i, \gamma_i)$ for $i = 1, 2, \ldots, t$, then $G^t \cong H$. It is easy to see that $\sim_\Gamma$ is a equivalence relation. If $G \sim_\Gamma H$, then we say G *is switch equivalent to* H *with respect to* Γ, or that G *is* Γ-*switchable to* H.

Some authors consider switch equivalence as being with respect to labelled graphs (see [15]). In this case only m-edge-coloured graphs with the same underlying graph can be switch equivalent. For example, if both G and H have underlying graph P_3, but $\{1, 3\}$ is independent in G while $\{2, 3\}$ is independent in H, then G and H can not be switch equivalent. Our definition coincides with this one, up to automorphisms of the underlying graph. Since our interest is in homomorphisms, taking automorphisms of the underlying graph into account is better suited to our purposes. Under our definition any two m-edge-coloured graphs with underlying graph P_3 are switch equivalent with respect to any group $\Gamma \leq S_m$ which acts transitively on $\{0, 1, \ldots, m-1\}$.

Let G and H be m-edge-coloured graphs. A *homomorphism of* G *to* H is a mapping of the vertices of G to the vertices of H that preserves the edges and edge sets. Formally, it is a function $h : V(G) \to V(H)$ such that $xy \in E_i(G)$ implies $f(x)f(y) \in E_i(H)$. We sometimes use the notation $h : G \to H$ to mean that h is a homomorphism of G to H; when the name of the function is unimportant the existence of a homomorphism of G to H is abbreviated to $G \to H$.

For $\Gamma \leq S_m$, a Γ-*switchable homomorphism* of G to H is a homomorphism $G' \to H$ for some G' which is switch equivalent to G with respect to Γ. If such a mapping exists then we say G is Γ-*switchably homomorphic to* H and write $G \to_\Gamma H$.

While the definition of a Γ-*switchable homomorphism* of G to H involves only switch equivalence to G, it transpires that switching at vertices of H can transform the question of whether $G \to_\Gamma H$ into a question about homomorphisms of m-edge-coloured graphs (without switching).

Theorem 2.1 ([10])**.** *Let G and H be m-edge-coloured graphs and $\Gamma \leq S_m$. Then $G \to_\Gamma H$ if and only if there exists $G' \sim_\Gamma G$ and $H' \sim_\Gamma H$ such that $G' \to H'$.*

In Section 5 we will determine, for all fixed m-edge-coloured graphs H and all $\Gamma \in \{S_m : m \geq 3\} \cup \{A_m : m \geq 4\} \cup \{D_m : m \geq 3\}$, the complexity deciding the existence of a Γ-switchable homomorphism from a given m-edge-coloured graph G to H. This is made possible by our work on switch-equivalence with respect to these groups in Sections 3 and 4.

A vertex k-colouring of a graph G can be regarded as a homomorphism of G to a complete graph K_k with vertex set $\{1, 2, \ldots, k\}$. More generally, a homomorphism of G to any k-vertex graph H can be regarded as a k-colouring of G: adjacent vertices in G are assigned adjacent, thus different, colours from $V(H)$.

For an integer $k \geq 1$, a *vertex k-colouring* of an m-edge-coloured graph G is a homomorphism of G to some m-edge coloured graph with k vertices. We note that, since non-adjacent vertices of H can be joined by an edge of any colour with destroying the existence of such a mapping, the underlying graph of H can be assumed to be complete. Put differently, a *vertex k-colouring* of an m-edge-coloured graph G is an assignment of k colours, say $1, 2, \ldots, k$, to the vertices of G such that adjacent vertices get different colours and the vertex colours respect the edge sets in the sense that each pair of different colours can appear on the ends of edges in at most one set E_i. It follows from the definition that vertices joined by a path of length two containing edges from different edge sets must be assigned different colours.

For $\Gamma \leq S_m$, a Γ-*switchable vertex k-colouring* of G is a Γ-switchable homomorphism $G \to_\Gamma H$, where $|V(H)| = k$. The complexity of deciding the existence of such a colouring has been determined for all Abelian groups:

Theorem 2.2 ([10]). *Let $\Gamma \leq S_m$ be Abelian. Then, the problem of deciding whether a given m-edge-coloured graph G has a Γ-switchable vertex k-colouring is Polynomial if $k = 2$, and NP-complete if $k \geq 3$.*

In Section 5 we will determine the complexity of deciding the existence of a Γ-switchable k-colouring when $\Gamma \in \{S_m : m \geq 3\} \cup \{A_m : m \geq 4\} \cup \{D_m : m \geq 3\}$.

The *chromatic number* of an m-edge-coloured graph G is the least k for which G has a vertex k-colouring, and is denoted by $\chi_e(G)$, or just χ_e when the context is clear. The Γ-*switchable chromatic number of* G is the least k for which G has a Γ-switchable vertex k-colouring, and is denoted by $\chi_\Gamma(G)$, or simply χ_Γ.

As an aside, we note that the chromatic number of an m-edge-coloured graph can differ substantially from that of its underlying graph. For example, consider the 2-edge-coloured complete bipartite graph with vertex set $V = \{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\}$ and edge sets $E_0 = \{a_i b_j : i \geq j\}$ and $E_1 = \{a_i b_j : i < j\}$. The underlying graph has chromatic number two, but $\chi_e = 2n = |V|$ because every two non-adjacent vertices are joined by a path of length two that uses an edge from E_0 and an edge from E_1.

When Γ is Abelian, there is simple relationship between χ_e and χ_Γ:

Theorem 2.3 ([10]). *Let $\Gamma \leq S_m$ be Abelian. For any m-edge-coloured graph G, $\chi_\Gamma(G) \leq \chi_e(G) \leq |\Gamma|\chi_\Gamma(G)$.*

When $m = 2$ switching occurs with respect to S_2. The upper bound on $\chi_e(G)$ is $2\chi_\Gamma(G)$, a result which appears in [13] (see [8] for the corresponding theorem for oriented graphs with the vertex pushing operation). In Section 5 we shall give evidence that no such simple relationship is likely to hold when Γ is non-Abelian.

3. Switching using groups with Property $\mathcal{T}_j$

In this section we describe a group property which implies the existence a sequence of switches which transforms the colour of given edge xy from i to j, and leaves the colour of all other edges unchanged.

When there exists j such that this property holds for all i, any m-edge-coloured graph G is Γ-switchable to one which is *monochromatic of colour* j (all edges have colour j). If Γ has this property for all i and j, then any two graphs with isomorphic underlying graphs are switch equivalent with respect to Γ. There is a sense in which the results in this section say that groups with property $\mathcal{T}_j$ are less interesting.

Let Γ be a group of permutations of $\{0, 1, \ldots, m-1\}$. For $i, j \in \{0, 1, \ldots, m-1\}$, we say Γ has *Property* $\mathcal{T}_{i,j}$ if it acts transitively on $\{0, 1, \ldots, m-1\}$ and there exists $k \in \{0, 1, \ldots, m-1\}$ and a permutation in $\alpha \in \mathit{Stabilizer}(k)$ such that $\alpha(i) = j$.

If $i \neq j$, then property $\mathcal{T}_{i,j}$ requires that some $k \in \{0, 1, \ldots, m-1\}$ has a non-trivial stabilizer (and so, by the Orbit-Stabilizer Theorem, every $k \in \{0, 1, \ldots, m-1\}$ does). Hence, and group with property $\mathcal{T}_{i,j}$ for some $i \neq j$, is necessarily non-Abelian. Further, it is implicit in the definition that any such group has order at last three.

Lemma 3.1. *Let G be a m-edge-coloured graph and $\Gamma \leq S_m$. If Γ has property $\mathcal{T}_{i,j}$ and the edge xy has colour i in G, then G is Γ-switchable to a m-edge-coloured graph G' in which xy has colour j and all other edges have the same colour as in G.*

Proof. Since Γ has Property $\mathcal{T}_{i,j}$, there exists $k \in \{0, 1, \ldots, m-1\}$ and $\alpha \in \mathit{Stabilizer}(k)$ such that $\alpha(i) = j$, and $\beta \in \Gamma$ such that $\beta(j) = k$. Consider the sequence of switches $(\alpha, x), (\beta, y), (\alpha^{-1}, x), (\beta^{-1}, y)$, which transform G into G'.

The only edges which change colour during this sequence of switches are those incident with x or y, After the first, second, third and fourth switch in the sequence, the edge xy has colour j, k, k, j, respectively. Any edge e incident with x and not y changes from its colour c_e to $\alpha(c_e)$ and then back to c_e. Similarly, any edge incident with y and not x changes from its colour and then back to it. The result follows. $\square$

For $j \in \{0, 1, \ldots, m-1\}$, we say that Γ has *Property* $\mathcal{T}_j$ if it has Property $\mathcal{T}_{i,j}$ for every $i \in \{0, 1, \ldots, m-1\}$.

Theorem 3.2. *If Γ has Property $\mathcal{T}_j$ for all $j \in \{0, 1, \ldots, m-1\}$, then any two m-edge-coloured graphs with isomorphic underlying graphs are switch equivalent with respect to Γ.*

Proof. Suppose G and H are m-edge-coloured graphs with isomorphic underlying graphs. Assume that the vertices of G have been relabelled so that the underling graph of G equals the underling graph of H. Define the *distance between* G *and* H to be the number of edges of G which have a different colour than the corresponding edge in H.

The proof is by induction on the distance between G and H. If the distance between G and H is zero, then they are equal and hence switch equivalent with respect to Γ. Suppose that any two m-edge-coloured graphs with isomorphic underlying graphs which are at distance less than d are switch equivalent with respect to Γ. Let G and H be m-edge-coloured graphs with isomorphic underlying graphs which are at distance d.

By Lemma 3.1, there is a sequence of switches that transform G into an m-edge-coloured graph G' which is at distance $d-1$ from H. By the induction hypothesis, G' is Γ-switchable to H. Therefore so is G. The result now follows by induction. □

Essentially the same argument as above proves the following.

Proposition 3.3. *Let G be a m-edge-coloured graph and $\Gamma \leq S_m$. If Γ has Property $\mathcal{T}_j$, then G is Γ-switchable to a m-edge-coloured graph G' which is monochromatic of colour j.*

Corollary 3.4. *Let G be a m-edge-coloured graph such that the underlying graph of G is bipartite. If $\Gamma \leq S_m$ has Property $\mathcal{T}_j$, then G is Γ-switchable to a m-edge-coloured graph G' which is monochromatic of colour i for any $i \in \{0, 1, \ldots, m-1\}$.*

Proof. Let (X, Y) be a bipartition of the underlying graph of G. By Proposition 3.3, G is Γ-switchable to a m-edge-coloured graph G' which is monochromatic of colour j. Since Γ acts transitively on $\{0, 1, \ldots, m-1\}$, for any $i \in \{0, 1, \ldots, m-1\}$, there is a permutation $\alpha \in \Gamma$ such that $\alpha(j) = i$. Switching with respect to α at each vertex in X transforms G' to a m-edge-coloured graph in which every edge has colour i. □

We observe that any 2-transitive group of permutations of $\{0, 1, \ldots, m-1\}$ has Property $\mathcal{T}_j$ for every $j \in \{0, 1, \ldots, m-1\}$. In particular, the symmetric group S_m is 2-transitive for all $m \geq 3$, and the Alternating group A_m is 2-transitive for all $m \geq 4$. By contrast, S_2, A_2 and A_3 are all Abelian. None of these have property $\mathcal{T}_{i,j}$ for $i \neq j$, and none can be used to switch any m-edge coloured graph to one where all edges have the same colour.

The following is now an easy consequence of Proposition 3.3.

Corollary 3.5. *Let $\Gamma \in \{S_m : m \geq 3\} \cup \{A_m : m \geq 4\}$. Any m-edge-coloured graph G is Γ-switchable to a m-edge-coloured graph G' which is monochromatic of colour j, for any $j \in \{0, 1, \ldots, m-1\}$.*

For $S_m,\ m \geq 3$, the corollary was independently proved by Christopher Duffy [private communication] using essentially the same argument as given above.

4. Switching with respect to Dihedral groups

Recall that we denote by D_m the group of symmetries of the regular m-gon with vertex set $\{0, 1, \ldots, m-1\}$. It transpires that the cases of m odd and m even are different. We consider the case of odd m first.

Proposition 4.1. *For any odd integer $m \geq 3$, the group D_m has Property $\mathcal{T}_j$ for every $j \in \{0, 1, \ldots, m-1\}$.*

Proof. Let $i \in \{0, 1, \ldots, m-1\}$. Since m is odd, either the least residue of $i - j$ modulo m is even, or the least residue of $j - i$ modulo m is even. Without loss of generality, the latter holds. Then there exists $k \in \{0, 1, \ldots, m-1\}$ such that $j - k \equiv k - i \pmod{m}$. Let α be the permutation of $\{0, 1, \ldots, m-1\}$ which corresponds to flipping the m-gon about vertex k. Then $\alpha \in \mathit{Stabilizer}(k)$ and $\alpha(i) = j$, so D_m has property $\mathcal{T}_{i,j}$ This completes the proof. $\square$

Corollary 4.2. *If $m \geq 3$ is odd, then any two m-edge-coloured graphs with isomorphic underlying graphs are switch equivalent with respect to D_m. In particular, any m-edge-coloured graph is D_m-switchable to one which is monochromatic of colour j, for any $j \in \{0, 1, \ldots, m-1\}$.*

We now turn to the case where m is even. Some basic results from group theory are needed. Note that if $m \geq 2$ is even and we let $\mathcal{E} = \{0, 2, \ldots, m-2\}$ and $\mathcal{O} = \{1, 3, \ldots, m-1\}$, then the partition $\{\mathcal{O}, \mathcal{E}\}$ is a block system for the action of D_m on $\{0, 1, \ldots, m-1\}$.

Proposition 4.3. *Suppose $m \geq 2$ is even, and let $\mathcal{E} = \{0, 2, \ldots, m-2\}$ and $\mathcal{O} = \{1, 3, \ldots, m-1\}$. Then,*

1. *$\mathit{Stabilizer}(\mathcal{E}) = \mathit{Stabilizer}(\mathcal{O})$ is a normal subgroup of D_m,*
2. *$D_m/\mathit{Stabilizer}(\mathcal{E}) \cong S_2$,*
3. *$\mathit{Stabilizer}(\mathcal{E})$ has Property $\mathcal{T}_{i,j}$ for all $i, j \in \mathcal{E}$, and*
4. *$\mathit{Stabilizer}(\mathcal{O})$ has Property $\mathcal{T}_{i,j}$ for all $i, j \in \mathcal{O}$.*

Proof. We prove only statement (3). The proof of (4) is similar. Let $i, j \in \mathcal{E}$. Without loss of generality, $i < j$. Since i and j are even, $k = (j-i)/2$ is an integer. Let α be the permutation corresponding to a flip of the regular m-gon about vertex k. Then $\alpha \in \mathit{Stabilizer}(k)$ and $\alpha(i) = j$. The result follows. $\square$

Let G be an m-edge coloured graph, where $m \geq 2$ is an even integer. The 2-edge coloured graph G_2 is obtained from G by assigning each edge its colour in G modulo 2.

Theorem 4.4. *Let G and H be m-edge-coloured graphs, where $m \geq 2$ is an even integer. Then G and H are switch equivalent with respect to D_m if and only if G_2 and H_2 are switch equivalent with respect to S_2.*

Proof. Suppose G and H are switch equivalent with respect to D_m. Then there is a sequence of switches $\mathcal{S} = (\pi_1, x_1), (\pi_2, x_2), \ldots, (\pi_t, x_t)$ that transforms G so it is isomorphic to H. Each permutation $\pi_i \in D_m$ either fixes both $\mathcal{E}$ and $\mathcal{O}$, or exchanges them. Let $\mathcal{S}'$ be the subsequence of $\mathcal{S}$ consisting of the permutations that exchange $\mathcal{E}$ and $\mathcal{O}$. Replacing each of the permutations in this subsequence by the transposition $(0\ 1)$ gives a sequence of switches that transforms G_2 so it is isomorphic to H_2.

Now suppose G_2 and H_2 are switch equivalent with respect to S_2. Then there is a sequence of switches $\mathcal{A} = (\sigma_1, x_1), (\sigma_2, x_2), \ldots, (\sigma_p, x_p)$ that transforms G_2 to a 2-edge-coloured graph G_2' which is isomorphic to H_2. Replacing each permutation $\sigma_i \in S_2$ by $(0\ 1\ \cdots\ m-1) \in D_m$ gives a sequence of switches that transforms G to a graph G' in which edge edge colour has the same parity as in G_2'. Since both $\mathit{Stabilizer}(\mathcal{E})$ and $\mathit{Stabilizer}(\mathcal{O})$ have Property $\mathcal{T}_{i,j}$ for all $i, j \in \mathcal{E}$, and all $i, j \in \mathcal{O}$, respectively, the m-edge-coloured graph G' is D_m-switchable to H. $\square$

Let G_2 be a 2-edge-coloured graph. Define the graph $P_{S_2}(G_2)$ to have vertex set $V(G_2) \times S_2$, with $(x, \pi_1)(y, \pi_2) \in E_i(P_{S_2}(G_2))$ if and only if $xy \in E_j(G_2)$ and $\pi_1\pi_2(j) = i$.

Brewster and Graves [2] (also see [8]) proved that the question of whether two m-edge-coloured graphs are switch equivalent with respect to $\mathbb{Z}_m$ can be translated into a question about isomorphism of m-edge-coloured graphs. A similar theorem holds for any Abelian group Γ [10]. We state the result only for $S_2 = \mathbb{Z}_2$.

Theorem 4.5 ([2]). *Let G_2 and H_2 be 2-edge-coloured graphs. Then G_2 and H_2 are switch equivalent with respect to S_2 if and only if $P_{S_2}(G_2) \cong P_{S_2}(H_2)$.*

Corollary 4.6. *Let G and H be m-edge-coloured graphs, where $m \geq 2$ is an even integer. Then G and H are switch equivalent with respect to D_m if and only if $P_{S_2}(G_2) \cong P_{S_2}(H_2)$.*

An analogue of the following theorem of Zaslavsky is immediate.

Theorem 4.7 ([14]). *The 2-edge-coloured graphs G and H with the same underling graph are switch equivalent with respect to $\mathbb{Z}_2$ if and only if some automorphism of the underlying graph transforms G to a 2-edge-coloured graph which the same collection of cycles with an odd number of edges in E_0 as H.*

Corollary 4.8. *Suppose $m \geq 2$ is even. The m-edge-coloured graphs G and H are switch equivalent with respect to D_m if and only if some automorphism of the underlying graph transforms G to an m-edge-coloured graph with the same collection of cycles with an odd number of edges whose colour is in $\mathcal{E}$.*

Proof. Since switching at a vertex preserves the parity of the number of edges on each cycle whose colour is in $\mathcal{E}$, any two m-edge-coloured graphs with the same underlying graph which are switch equivalent with respect to D_m have the same collection of cycles with an odd number of edges whose colour is in $\mathcal{E}$. Hence if G and H are switch equivalent with respect to D_m then some automorphism of the underlying graph transforms G to an m-edge-coloured graph with the same collection of cycles with an odd number of edges whose colour is in $\mathcal{E}$.

Suppose some automorphism of the underlying graph transforms G to an m-edge-coloured graph with the same collection of cycles with an odd number of edges whose colour is in $\mathcal{E}$. Then, by Theorem 4.7, the corresponding 2-edge-coloured graphs G_2 and H_2 are switch equivalent with respect to S_2. The result now follows from Theorem 4.4. $\square$

Corollary 4.9. *Suppose $m \geq 2$ is even. Then,*

(1) *for any $i \in \mathcal{E}$, the m-edge-coloured graph G is D_m-switchable to a m-edge-coloured graph G' which is monochromatic of colour i if and only if every cycle in G has an even number of edges whose colour is in $\mathcal{O}$, and*
(2) *for any $j \in \mathcal{O}$, the m-edge-coloured graph G is D_m-switchable to a m-edge-coloured graph G' which is monochromatic of colour*

j if and only if every cycle in G has an even number of edges whose colour is in $\mathcal{E}$.

Corollary 4.10. *Suppose $m \geq 2$ is even. If the underlying graph of the m-edge-coloured graph G is bipartite then, for any $j \in \{0, 1, \ldots, m-1\}$, G is D_m-switchable to a graph which is monochromatic of colour j if and only if every cycle has an even number of edges whose colour is in $\mathcal{O}$.*

5. Switchable homomorphisms and vertex colourings

For a fixed m-edge-coloured graph H and $\Gamma \leq S_m$, we define Γ-*switchable Hom$_H$* to be the problem of deciding whether there is a Γ-switchable homomorphism of G to H. In what follows we will determine the complexity of Γ-switchable Hom$_H$ for all groups Γ with property $\mathcal{T}_j$, and then D_m for even $m \geq 2$.

We will make use of the following.

Theorem 5.1 ([6])**.** *Let H be a finite, undirected graph. If H is bipartite, then Hom$_H$ is polynomial. If H is not bipartite, then Hom$_H$ is NP-complete.*

Theorem 5.2. *Let $\Gamma \leq S_m$ have Property $\mathcal{T}_j$, and let H be an m-edge-coloured graph. If the underlying graph of H is bipartite, then Γ-switchable Hom$_H$ is Polynomial. If the underlying graph of H is not bipartite, then Γ-switchable Hom$_H$ is NP-complete.*

Proof. Let H' be the underlying graph of H. For a given m-edge-coloured graph G, we claim that $G \to_\Gamma H$ if and only if $G' \to H'$. It is clear that if $G \to_\Gamma H$, then $G' \to H'$. Suppose $G' \to H'$. Both G and H are Γ-switchable to m-edge-coloured graphs G_j and H_j, respectively, which are monochromatic of colour j. Since $G_j \to H_j$, the claim follows from Theorem 2.1.

Suppose that H' is bipartite. By the claim, if G is an m-edge-coloured graph, then $G \to_\Gamma H$ if and only if the underlying graph of G is bipartite. It follows that Γ-switchable Hom$_H$ is Polynomial.

Now suppose that H' is not bipartite. The transformation is from Hom$'_H$. Let G' be an instance of Hom$'_H$, that is, an undirected graph. The transformed instance of Γ-switchable Hom$_H$ is the m-edge-coloured graph G which is monochromatic of colour j and has underlying graph G'. The transformation can clearly be carried out in polynomial time. The result now follows from the claim, and Theorem 5.1. $\square$

Corollary 5.3. *Let $\Gamma \in \{S_m : m \geq 3\} \cup \{A_m : m \geq 4\} \cup \{D_m : \text{odd } m \geq 3\}$, and let H be a m-edge-coloured graph. If the underlying graph of H is bipartite, then Γ-switchable Hom_H is Polynomial. If the underlying graph of H is not bipartite, then Γ-switchable Hom_H is NP-complete.*

It remains to consider the dihedral groups D_m where $m \geq 2$ is even. We will use the following special case of a theorem of Brewster et al. [3]. Note that the property that every cycle has an even number of edges of each colour can hold only if the underlying graph is bipartite.

Theorem 5.4 ([3])**.** *Let H be a 2-edge-coloured-graph. If every cycle has an even number of edges of each colour, then S_2-switchable Hom_H is Polynomial. If H has a cycle with an odd number of edges of some colour, then S_2-switchable Hom_H is NP-complete.*

The existence of a S_2-switchable homomorphism between two 2-edge-coloued graphs can be transformed into one about the existence of a homomorphism without switching. A similar statement holds for all Abelian groups, see [10].

Theorem 5.5 ([2, 8])**.** *Let G_2 and H_2 be 2-edge-coloured graphs. Then $G_2 \to_{S_2} H_2$ if and only if $P_{S_2}(G_2) \to P_{S_2}(H_2)$.*

Corollary 5.6. *Let H be a m-edge-coloured graph, where $m \geq 2$ is even. If every cycle has an even number of edges whose colour is in $\mathcal{E}$ and an even number of edges whose colour is in $\mathcal{O}$, then D_m-switchable Hom_H is Polynomial. If H has a cycle with an odd number of edges whose colour is in $\mathcal{E}$ or an odd number of edges whose colour is in $\mathcal{O}$, then D_m-switchable Hom_H is NP-complete.*

Proof. Suppose If every cycle in H has an even number of edges whose colour is in $\mathcal{E}$ and and an even number of edges whose colour is in $\mathcal{O}$. Then every cycle in H_2 has an even number of edges of each colour. By Theorem 5.4, S_2-switchable Hom_{H_2} is Polynomial. It follows from Theorem 4.4 that D_m-switchable Hom_H is Polynomial.

Now suppose H has a cycle with an odd number of edges whose colour is in $\mathcal{E}$ or an odd number of edges whose colour is in $\mathcal{O}$. The transformation is from S_2-switchable Hom_{H_2}. Let G_2 be an instance of S_2-switchable Hom_{H_2}. The transformed instance of D_m-switchable Hom_H is the m-edge-coloured graph G which is identical to G_2 (the edge sets $E_2, E_3, \ldots, E_{m-1}$ are all empty). The transformation can clearly be carried out in Polynomial time. The result follows from Theorem 4.4. □

We now determine the complexity of deciding whether a given m edge-coloured graph G has a Γ-switchable vertex k-colouring for the groups Γ we have been considering.

Theorem 5.7. *Let $k \geq 2$ be in integer and $\Gamma \leq S_m$. If $\Gamma \in \{S_m : m \geq 2\} \cup \{A_m : m \geq 2\} \cup \{D_m : m \geq 2\}$ or if Γ has property $\mathcal{T}_j$ for some j, then the problem of deciding whether a given m-edge-coloured graph G is Γ-switchably k-colourable is Polynomial if $k = 2$, and NP-complete if $k \geq 3$.*

Proof. If $\Gamma \in \{S_2, S_2, A_3, D_2\}$ then the statement follows from Theorem 2.2. Hence suppose $\Gamma \in \{S_m : m \geq 3\} \cup \{A_m : m \geq 4\} \cup \{D_m : m \geq 3\}$.

An m-edge-coloured graph G has a vertex 2-colouring if and only if it is bipartite and Γ-switchable to be monochromatic of some colour. That is, if there is a Γ-switchable homomorphism to a monochromatic K_2. By Theorem 5.2 and Corollary 5.6, this can be decided in Polynomial time.

Suppose $k \geq 3$ and $\Gamma \leq S_m$ has Property $\mathcal{T}_j$. The transformation is from vertex k-colouring of undirected graphs. Recall that a vertex k-colouring of an undirected graph G is a homomorphism $G \to K_k$. Suppose an undirected graph G is given. The transformed instance is the m-edge-coloured graph G_j which is monochromatic of colour j. The transformation can be accomplished in Polynomial time. A vertex k-colouring of G is clearly a Γ-switchable vertex k-colouring of G_j, and vice-versa.

Finally, suppose $k \geq 3$ and $\Gamma \in \{D_m : m \geq 4 \text{ } \textit{even}\}$. The transformation is from S_2-switchable vertex k-colouring. Let G_2 be an instance of S_2-switchable vertex k-colouring (which is NP-complete by Theorem 2.2). The transformed instance of D_m-switchable vertex k-colouring is the m-edge-coloured graph G which is identical to G_2 (the edge sets $E_2, E_3, \ldots, E_{m-1}$ are all empty). The transformation can clearly be carried out in Polynomial time. The result follows from Theorem 4.4. $\square$

If Γ has property T_j for some $j \in \{0, 1, \ldots, m-1\}$, then a certificate that a given m-edge-coloured graph G does not have a vertex 2-colouring is an odd cycle in the underlying graph of G. If Γ is D_m, where $m \geq 2$ is even, then it is a cycle with an odd number of edges in $\mathcal{E}$ or an odd number of edges in $\mathcal{O}$.

We conclude the paper by considering whether there might be a version of Theorem 2.3 for non-Abelian groups. Our goal is to suggest that the constant in the upper bound is likely to be a fairly complicated function of the group. We will make use of a result for m-edge-coloured graphs which is similar to Observations 1 and 4 in [9] (for oriented graphs).

Proposition 5.8. *Let G be a graph, and k be a positive integer. If $\chi_e \leq k$ for every m-edge-colouring of G, then $m^{\binom{k}{2}} k^{|V|} \geq m^{|E|}$.*

Proof. Regard the vertices of G as being labelled. Then there are $m^{|E|}$ labelled m-edge-coloured graphs with underlying graph G.

Any partition of the vertex set of G into at most k independent sets is a vertex k-colouring of at most $m^{\binom{k}{2}}$ of the $m^{|E|}$ labelled m-edge-coloured graphs with underlying graph G. Since the number of possible vertex k-colourings is less than $k^{|V|}$, it follows that $m^{\binom{k}{2}} k^{|V|} \geq m^{|E|}$. □

Proposition 5.9. *For every integer $\Delta \geq 2$, there exists an m-edge-coloured graph for which the underlying graph is Δ-regular and $\chi_e > m^{\Delta/2}$.*

Proof. Let G be a Δ-regular graph. Then G has $\Delta|V|/2$ edges. By Proposition 5.8, if every m-edge-colouring of G has $\chi_e \leq m^{\Delta/2}$, then $\log_m(\chi_e) \geq \Delta/2 - \binom{m^{\Delta/2}}{2}/|V|$, or equivalently, $\chi_e \geq m^{\Delta/2} \cdot m^{|V|/\binom{m^{\Delta/2}}{2}}$.

Since there are Δ-regular graphs with arbitrarily many vertices, and m and Δ are constants, the graph G can be chosen to have enough vertices so that

$$m^{|V|/\binom{m^{\Delta/2}}{2}} > 1.$$

It follows that some m-edge-colouring of G has $\chi_e > m^{\Delta/2}$. □

Suppose we seek a version of Theorem 2.3 for D_m, where m is odd. Since $\chi_{D_m}(G) = \chi(U)$, where U is the underlying graph of G, we would need a constant c such that

$$m^{\Delta/2} < \chi_e \leq c \cdot \chi(U) \leq c(1 + \Delta(U)) \leq m\Delta(U)m^{\Delta+1},$$

where the Brooks-like upper bound is from [10]. Such a constant c must be exponential in the maximum degree of the underlying graph, whereas $|D_m|$ is linear in m.

We suggest that the underlying reason is that for Abelian groups it can be assumed that there is one switch at each vertex (possibly with respect to the identity element), whereas for non-Abelian groups the order in which the switches are made is important.

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